



Lecture 03

Fundamentals of Business Mathematics

MATH-114 • BBA 2k26

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Lecture 03 Outline: Systems of Linear Equations & Multi-Variable Modeling

Part 1: Linear Equations in > 2 Variables

- n -Variable General Equations
- 3D Coordinate Systems & Octants
- Intercepts & Plane Geometry
- Special Planes & Missing Variables

Part 2: Multi-Variable Business Applications

- Resource Allocation Constraints
- Emergency Airlift Volume Model
- Court Docket Time Capacity
- Multi-Product Production Limits

Part 3: Systems of Linear Equations

- System Dimensions ($m \times n$)
- Graphical Solution Sets (2×2 Systems)
- Unique, Inconsistent, & Dependent Systems
- System Classification via Slope Analysis

Part 4: Algebraic Elimination & Overdetermined Systems

- The Elimination Procedure
- Identifying Algebraic Signals ($0 = 0$ vs. $0 = k$)
- Overdetermined Systems ($m \times 2, m > 2$)
- Solving & Testing Consistency



Linear Equations in More Than Two Variables

Algebraic Structure of Multi-Variable Linear Equations

Linear Equation in n Variables

A linear equation involving n variables $x_1, x_2, \dots, x_n$ is expressed in the standard form:

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

where $a_1, a_2, \dots, a_n$ are constant coefficients, b is the constant term, and not all $a_i = 0$.



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- **Three-Variable Linear Equation:** Setting $n = 3$ yields:

$$a_1x_1 + a_2x_2 + a_3x_3 = b \quad \text{or} \quad ax + by + cz = d$$

- **Key Properties:**

- ▶ Every variable is raised strictly to the first power (x_i^1).
- ▶ No products of variables (e.g., x_1x_2 , yz) exist.
- ▶ Solution set consists of ordered n -tuples $(x_1, x_2, \dots, x_n)$ satisfying the equation.



3D Coordinate Space & Octants

The 3D Rectangular Coordinate System:

- Built from three mutually perpendicular axes: x_1 , x_2 , and x_3 .
- The axes intersect at the **Origin** $(0, 0, 0)$.
- The axes define three coordinate planes (x_1x_2 , x_1x_3 , x_2x_3), dividing 3-space into **eight octants**.



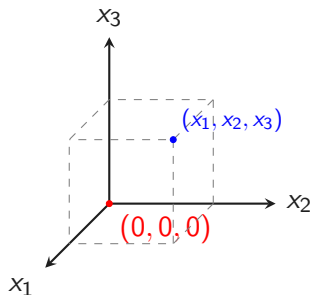
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Economic Domain Constraint:

In business applications, non-negativity constraints ($x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$) restrict feasible solutions to the **First Octant**.



Plane Geometry & Coordinate Intercepts

Geometric Interpretation of 3-Variable Linear Equations:

While a 2-variable linear equation graphs as a straight line in 2-space, a 3-variable linear equation $a_1x_1 + a_2x_2 + a_3x_3 = b$ graphs as a **plane** in 3-space.



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Determining Planes via Coordinate Intercepts:

- Three non-collinear points uniquely define a plane.
- **x_1 -intercept:** Set $x_2 = 0, x_3 = 0 \implies (x_1, 0, 0)$.
- **x_2 -intercept:** Set $x_1 = 0, x_3 = 0 \implies (0, x_2, 0)$.
- **x_3 -intercept:** Set $x_1 = 0, x_2 = 0 \implies (0, 0, x_3)$.



Example: Graphing a 3-Variable Plane

Problem: Graph the linear plane

$$2x_1 + 4x_2 + 3x_3 = 12.$$



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Find Coordinate Intercepts:

- **x_1 -intercept:** Set $x_2 = 0, x_3 = 0$:

$$2x_1 = 12 \implies x_1 = 6 \implies (6, 0, 0)$$

- **x_2 -intercept:** Set $x_1 = 0, x_3 = 0$:

$$4x_2 = 12 \implies x_2 = 3 \implies (0, 3, 0)$$

- **x_3 -intercept:** Set $x_1 = 0, x_2 = 0$:

$$3x_3 = 12 \implies x_3 = 4 \implies (0, 0, 4)$$



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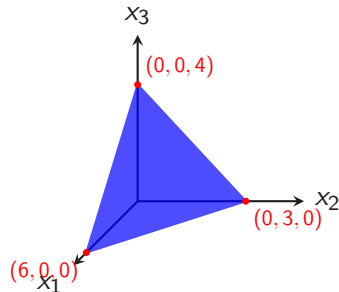
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Special Planes & Missing Variables

One Missing Variable ($a_1x_1 + a_2x_2 = b$):

- Graphs as a plane **parallel to the axis of the missing variable**.
- *Example:* $2x_1 + 4x_2 = 12$ graphs as a plane parallel to the x_3 -axis.



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Two Missing Variables ($x_1 = k$):

- Graphs as a plane **parallel to the coordinate plane** of the two missing variables.
- *Example 21:* $x_1 = 0$ represents the entire x_2x_3 coordinate plane.
- *Example:* $x_3 = 5$ is a horizontal plane 5 units above the x_1x_2 floor.



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Hyperplanes ($n > 3$): For $n \geq 4$ variables, linear equations graph as **hyperplanes** in n -dimensional space. While impossible to visualize geometrically, their algebraic properties remain identical!



Multi-Variable Business Applications

Resource Allocation & Multi-Item Constraints

Commercial Modeling with Multiple Decision Variables:

In enterprise operations, managers rarely handle just one or two products. Single linear equations frequently model total resource consumption across n operational activities:

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = B$$

where c_i represents resource usage per unit of activity i , x_i is the volume of activity i , and B is total available budget/capacity.



Example: Emergency Airlift Logistics

Problem Context: A relief agency organizes an emergency airlift to a disaster zone. Four cargo items are loaded into aircraft with a maximum volume capacity of **6,000 cubic feet** per flight.



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Cargo Category	Item 1 (x_1)	Item 2 (x_2)	Item 3 (x_3)	Item 4 (x_4)
Volume per Container (ft^3)	2.0	3.0	1.5	4.0



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Volume per Container (ft ³)	2.0	3.0	1.5	4.0

Linear Capacity Equation:

$$2x_1 + 3x_2 + 1.5x_3 + 4x_4 = 6000$$

where $x_i \geq 0$ represents the number of containers of item i loaded.



Example: Court Docket Time Scheduling

Legal Operations Model: A district court judge has **850 hours** of trial time available over a term to adjudicate three categories of civil cases:

- Case Type 1 (x_1): Requires **16 trial hours** per case.
- Case Type 2 (x_2): Requires **8 trial hours** per case.
- Case Type 3 (x_3): Requires **4.5 trial hours** per case.

$$16x_1 + 8x_2 + 4.5x_3 = 850$$



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$$16x_1 + 8x_2 + 4.5x_3 = 850$$

- **Single-Case Limits (Maximum Capacity per Type):**

- ▶ Max Type 1 ($x_2 = 0, x_3 = 0$): $16x_1 = 850 \implies x_1 = \mathbf{53.125}$ cases $\implies$ 53 full cases.
- ▶ Max Type 2 ($x_1 = 0, x_3 = 0$): $8x_2 = 850 \implies x_2 = \mathbf{106.25}$ cases $\implies$ 106 full cases.
- ▶ Max Type 3 ($x_1 = 0, x_2 = 0$): $4.5x_3 = 850 \implies x_3 = \mathbf{188.89}$ cases $\implies$ 188 full cases.



Applied Case: Multi-Product Manufacturing Mix

Problem Statement: A manufacturer allocates **240 total labor-hours** across three product lines: Product A (x_1 , 5 hrs/unit), Product B (x_2 , 3.5 hrs/unit), Product C (x_3 , 7.5 hrs/unit).



Applied Case: Multi-Product Manufacturing Mix

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Labor Constraint Model:

$$5x_1 + 3.5x_2 + 7.5x_3 = 240$$



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Operational Capacity Analysis:

- ① If 20 units of Product A ($x_1 = 20$) and 10 units of Product C ($x_3 = 10$) are scheduled:

$$5(20) + 3.5x_2 + 7.5(10) = 240 \implies 100 + 3.5x_2 + 75 = 240$$

$$3.5x_2 + 175 = 240 \implies 3.5x_2 = 65 \implies x_2 = \mathbf{18.57} \text{ units of B}$$

- ② Maximum single-product production capacities:

$$\text{Max A} = 48 \text{ units}, \quad \text{Max B} = 68.57 \text{ units}, \quad \text{Max C} = 32 \text{ units}$$



Multi-Variable Resource Allocations

Discussion

Discussion Question: Discrete Constraints vs. Continuous Capacity

Consider the production equation $5x_1 + 3.5x_2 + 7.5x_3 = 240$:

- 1 In Scenario 1 above, $x_2 = 18.57$ units of Product B. If Product B represents discrete industrial generators, how should production be scheduled? What happens to the remaining 2 labor-hours?
- 2 What is the marginal trade-off rate between Product A and Product C ($\frac{\Delta x_3}{\Delta x_1}$)?



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Key Takeaway:

With fractional discrete outputs (18.57), real-world managers round down to **18 full units** ($3.5 \times 18 = 63$ hrs), leaving 2 hours of unallocated idle capacity or maintenance time!



Systems of Linear Equations & Graphical Analysis

Introduction to Linear Systems

Definition of an $m \times n$ System of Linear Equations

A system of linear equations consists of m linear equations involving n variables simultaneously:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$



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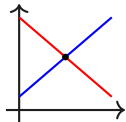
- **Dimension Notation:** An $m \times n$ **system** has m equations and n variables.
- **Solution to a System:** An ordered tuple $(x_1^*, x_2^*, \dots, x_n^*)$ that satisfies **all** m equations simultaneously.



Three Solution Possibilities for 2×2 Systems

For any 2×2 system of linear equations (two lines in 2-space), exactly **one** of three geometric conditions must hold:

1. Unique Solution



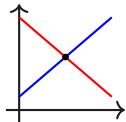
- **Intersecting Lines**
- Slopes differ ($m_1 \neq m_2$).
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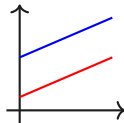
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2. No Solution



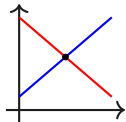
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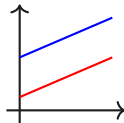
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- **Parallel Lines**
- Equal slopes ($m_1 = m_2$), different intercepts ($k_1 \neq k_2$).
- **Inconsistent System.**

3. Infinitely Many



- **Coincident Lines**
- Equal slopes ($m_1 = m_2$) & equal intercepts ($k_1 = k_2$).
- **Dependent System.**



Example: Graphical Solution of a 2×2 System

Solve Graphically:

$$\begin{cases} (1) & 2x + 4y = 20 \\ (2) & 3x + y = 10 \end{cases}$$



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Intersection Verification:

Substitute ($x^* = 2, y^* = 4$):

$$2(2) + 4(4) = 20 \quad \checkmark \quad | \quad 3(2) + 4 = 10 \quad \checkmark$$

Solution Set: **Unique Point** (2, 4).



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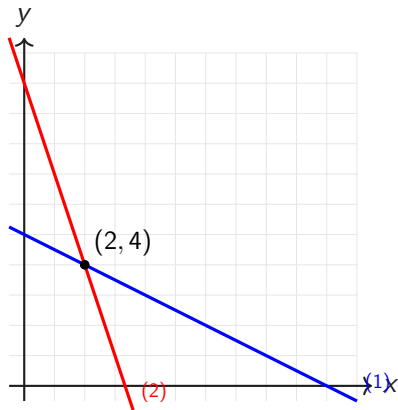
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Slope-Intercept Comparison Method

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$m_1 = m_2$	$k_1 \neq k_2$	Inconsistent	No Solution (Parallel)
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Efficiency Tip: Always check slopes first! If $m_1 \neq m_2$, you are guaranteed a unique solution before spending time performing elimination.



Algebraic Elimination & Overdetermined Systems

The Algebraic Elimination Procedure

Purpose of Algebraic Elimination:

Graphical solutions are limited by plotting precision. The elimination procedure uses linear combinations to systematically eliminate variables until a single-variable equation remains.



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Systematic Execution Steps:

- 1 Multiply equation(s) by non-zero constants so coefficients of one variable are equal in magnitude but opposite in sign.
- 2 Add the equations together to eliminate that variable.
- 3 Solve the resulting single-variable equation.
- 4 Back-substitute the solution into an original equation to find the second variable.



Example: Unique Solution via Elimination

Solve System:
$$\begin{cases} (1) & 2x + 4y = 20 \\ (2) & 3x + y = 10 \end{cases}$$



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Step-by-Step Elimination:

- ① Multiply Equation (2) by -4 :

$$-4(3x + y = 10) \implies -12x - 4y = -40 \quad (3)$$



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- ② Add Equation (1) and Equation (3):

$$(2x + 4y) + (-12x - 4y) = 20 + (-40) \implies -10x = -20 \implies x = 2$$



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- ② Add Equation (1) and Equation (3):

$$(2x + 4y) + (-12x - 4y) = 20 + (-40) \implies -10x = -20 \implies x = 2$$

- ③ Back-substitute $x = 2$ into Equation (2):

$$3(2) + y = 10 \implies 6 + y = 10 \implies y = 4$$

Final Solution: Unique Point $(2, 4)$.



Example: Dependent Systems (Infinitely Many)

Solve System:
$$\begin{cases} (1) & 3x - 2y = 6 \\ (2) & -15x + 10y = -30 \end{cases}$$



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Elimination Attempt:

- ① Multiply Equation (1) by 5:

$$5(3x - 2y = 6) \implies 15x - 10y = 30 \quad (3)$$



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- ② Add Equation (2) and Equation (3):

$$(-15x + 10y) + (15x - 10y) = -30 + 30 \implies 0 = 0 \quad \textbf{(Algebraic Identity!)}$$



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- ② Add Equation (2) and Equation (3):

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Interpretation: The result $0 = 0$ indicates the equations are **dependent** (coincident lines). Solution set: $S = \{(x, y) \mid y = \frac{3}{2}x - 3\}$.

Example: Inconsistent Systems (No Solution)

Solve System:
$$\begin{cases} (1) & 6x - 12y = 24 \\ (2) & -1.5x + 3y = 9 \end{cases}$$



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Elimination Attempt:

- ① Multiply Equation (2) by 4:

$$4(-1.5x + 3y = 9) \implies -6x + 12y = 36 \quad (3)$$



Example: Inconsistent Systems (No Solution)

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- ② Add Equation (1) and Equation (3):

$$(6x - 12y) + (-6x + 12y) = 24 + 36 \implies 0 = 60 \quad \textbf{(Algebraic Contradiction!)}$$



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- ② Add Equation (1) and Equation (3):

$$(6x - 12y) + (-6x + 12y) = 24 + 36 \implies 0 = 60 \quad \textbf{(Algebraic Contradiction!)}$$

Interpretation: The contradiction $0 = 60$ indicates the system is **inconsistent** (parallel lines).
Solution set: **Empty Set** ($\emptyset$).

Overdetermined Systems ($m \times 2, m > 2$)

Definition of Overdetermined Systems:

An **overdetermined system** has more equations than variables ($m > n$, e.g., 3×2 or 4×2). Graphically, this represents 3 or more lines competing to intersect at a single mutual point in 2-space.



Overdetermined Systems ($m \times 2, m > 2$)

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Standard Solution Protocol:

- 1 Select any **two** equations from the system and solve for (x^*, y^*) .
- 2 If no solution exists for those two, the entire system has no solution.
- 3 If a unique solution (x^*, y^*) is found, **test it in all remaining $m - 2$ equations**.
- 4 If (x^*, y^*) satisfies **every** remaining equation, it is the unique solution; if it fails in even one equation, the system has **no solution**.



Example: Solving a 4×2 Overdetermined System

Analyze System: $\begin{cases} (1) & x + 2y = 8, & (2) & 2x - 3y = -5 \\ (3) & -5x + 6y = 8, & (4) & x + y = 7 \end{cases}$



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Execution Procedure:

- ① Solve (1) & (2) simultaneously:

$$-2(x + 2y = 8) \implies -2x - 4y = -16$$

$$(-2x - 4y) + (2x - 3y) = -21 \implies -7y = -21 \implies y = 3, x = 2 \implies (2, 3)$$



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- ② Test (2, 3) in Equation (3): $-5(2) + 6(3) = -10 + 18 = 8 \quad \checkmark$ (Satisfied!).

- ③ Test (2, 3) in Equation (4): $2 + 3 = 5 \neq 7 \quad \times$ (**Contradiction!**).

Conclusion: The system is **inconsistent (No Solution)**.



Business Context: Over-Constrained Decision Making

Discussion

Real-World Meaning of Overdetermined Systems

Why do overdetermined systems arise in corporate planning?

- Each equation represents an independent operational constraint (labor hours, raw material, budget limit, policy target).
- Having $m > n$ means management has specified **more constraints than available operational levers**.



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Managerial Takeaway:

When an overdetermined business system yields "No Solution," management must either:

- 1 Relax one or more constraints (e.g., expand budget/hours).
- 2 Use optimization techniques (Linear Programming) to find the "best feasible" compromise.



Summary & Homework Assignment

Summary & Recommended Homework

Lecture 03 Summary

- n -Variable linear equations represent hyperplanes; 3D systems graph as planes in 8 octants.
- Multi-item business models use single linear equations to capture total capacity limits.
- Linear systems ($m \times n$) yield unique, infinite, or zero solutions.
- Overdetermined systems require verifying candidate solutions across all remaining constraints.



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Practice Homework (Budnick, 4th Ed)

- **Section 2.5:** Problems 1, 2, 3 (pp. 72–73)
- **Section 2.6:** Exercises 1, 7, 15 (pp. 76–78)
- **Section 3.1:**
 - ▶ Slope check: Ex 1–10 (p. 96)
 - ▶ Graphical: Ex 11–15 (p. 96)
 - ▶ Elimination: Ex 21–29 (p. 97)
 - ▶ Overdetermined: Ex 31–34 (p. 97)

Next Lecture: **Gaussian Elimination & Matrix Methods!**

