



Lecture 04

Fundamentals of Business Mathematics

MATH-114 • BBA 2k26

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Lecture 04 Outline: Gaussian Elimination & Business Systems

Part 1: Gaussian Elimination (2×2 Systems)

- Matrix Notation & Augmented Arrays
- The 3 Elementary Row Operations
- Target Reduced Diagonal Form
- Unique, Dependent, & Inconsistent Systems

Part 2: n -Variable Systems & 3×3 Reduction

- 3D Plane Geometry of 3×3 Systems
- Full Diagonalization Procedure
- Free Variables & Parameterized Solutions
- Underdetermined Systems ($m < n$)

Part 3: Real-World Business Applications

- Departmental Production Allocation
- Commercial Coffee Blending Model
- Portfolio Investment Asset Allocation
- Sensitivity Analysis

Part 4: Summary & Homework

- Key Methodological Summary
- Assigned Homework Exercises
- Preparation for Quiz 1



Gaussian Elimination for 2×2 Systems

Why Transition to Matrix Methods?

The Need for Systematic Matrix Algorithms

While graphical and substitution methods work well for two variables, commercial business problems routinely involve dozens of products, departments, and financial constraints.

- **Eliminates Variable Confusion:** Keeps coefficients organized without writing variable symbols repeatedly.



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- **Eliminates Variable Confusion:** Keeps coefficients organized without writing variable symbols repeatedly.
- **Algorithm-Driven:** Follows an exact, step-by-step reduction path suitable for computer optimization software.
- **Scalable:** Handles 3, 4, or n simultaneous linear equations systematically.



Augmented Matrix Notation

System to Augmented Matrix Representation

A system of linear equations:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

is written in compact augmented matrix format as:

$$\left(\begin{array}{cc|c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array} \right)$$

- The vertical line separates coefficient matrix A from constant vector B .



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- The vertical line separates coefficient matrix A from constant vector B .
- Column 1 corresponds to variable x , Column 2 to variable y , and Column 3 to constants.



The Three Elementary Row Operations

Permissible Operations on Augmented Matrices

Elementary row operations alter the appearance of the matrix without changing its underlying solution set:



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- ❸ **Operation 3 (Row Interchange):** Swap the positions of two rows ($R_i \leftrightarrow R_j$).



Target Reduced Diagonal Form & Pivoting Mechanics

Target Transformation for 2×2 Systems:

$$\left(\begin{array}{cc|c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array} \right) \xrightarrow{\text{Gaussian Elimination}} \left(\begin{array}{cc|c} \mathbf{1} & \mathbf{0} & v_1 \\ \mathbf{0} & \mathbf{1} & v_2 \end{array} \right)$$



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- ➊ **Step 1:** Transform element $(1, 1)$ into a **leading 1** using row scaling.



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- ➍ **Step 4:** Use row addition to convert element $(1, 2)$ into **0** (clear Column 2).



Example 1: Unique Solution System

Problem Statement: Solve the system using Gaussian Elimination:

$$\begin{cases} 5x + 20y = 25 \\ 4x - 7y = -26 \end{cases} \implies \left(\begin{array}{cc|c} 5 & 20 & 25 \\ 4 & -7 & -26 \end{array} \right)$$



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Step 1: Create Leading 1 in Row 1 ($R_1 \div 5 \rightarrow R_1$)

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Step 2: Clear Column 1 below Pivot ($-4R_1 + R_2 \rightarrow R_2$)

$$-4(1, 4 | 5) + (4, -7 | -26) = (0, -23 | -46) \implies \left(\begin{array}{cc|c} 1 & 4 & 5 \\ 0 & -23 & -46 \end{array} \right)$$



Example 1: Continued

Current State:

$$\left(\begin{array}{cc|c} 1 & 4 & 5 \\ 0 & -23 & -46 \end{array} \right)$$



Example 1: Continued

Current State:

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Step 3: Create Leading 1 in Row 2 ($R_2 \div (-23) \rightarrow R_2$)

$$\left(\begin{array}{cc|c} 1 & 4 & 5 \\ 0 & 1 & 2 \end{array} \right)$$



Example 1: Continued

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Step 4: Clear Column 2 above Pivot ($-4R_2 + R_1 \rightarrow R_1$)

$$-4(0, 1 \mid 2) + (1, 4 \mid 5) = (1, 0 \mid -3) \implies \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array} \right)$$



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Final Solution: $1x + 0y = -3 \implies x = -3$ and $0x + 1y = 2 \implies y = 2$.
Ordered Pair: $(-3, 2)$.



Example 2: Dependent System (Infinite Solutions)

Problem Statement: Solve using Gaussian Elimination:

$$\begin{cases} 3x - 2y = 6 \\ -15x + 10y = -30 \end{cases} \Rightarrow \left(\begin{array}{cc|c} 3 & -2 & 6 \\ -15 & 10 & -30 \end{array} \right)$$



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Step 2: Clear Column 1 ($15R_1 + R_2 \rightarrow R_2$)

$$15 \left(1, -\frac{2}{3} \mid 2 \right) + (-15, 10 \mid -30) = (0, 0 \mid 0) \Rightarrow \left(\begin{array}{cc|c} 1 & -\frac{2}{3} & 2 \\ 0 & 0 & 0 \end{array} \right)$$



Expressing Infinitely Many Solutions

Interpreting Row of Zeros ($0, 0 \mid 0$):

The equation $0x + 0y = 0$ is an identity ($0 = 0$), indicating that the two equations are dependent (coincident lines).



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Parametric Solution Set:

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- Let y be any arbitrary real number ($y \in \mathbb{R}$).
- Solution set:

$$S = \left\{ (x, y) \mid x = \frac{2}{3}y + 2, y \in \mathbb{R} \right\}$$



Example 3: Inconsistent System (No Solution)

Problem Statement: Solve using Gaussian Elimination:

$$\begin{cases} 6x - 12y = 24 \\ -1.5x + 3y = 9 \end{cases} \implies \left(\begin{array}{cc|c} 6 & -12 & 24 \\ -1.5 & 3 & 9 \end{array} \right)$$



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Step 1: Scale Row 1 ($R_1 \div 6 \rightarrow R_1$)

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$$\left(\begin{array}{cc|c} 1 & -2 & 4 \\ -1.5 & 3 & 9 \end{array} \right)$$

Step 2: Clear Column 1 ($1.5R_1 + R_2 \rightarrow R_2$)

$$1.5(1, -2 | 4) + (-1.5, 3 | 9) = (0, 0 | 15) \implies \left(\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 0 & 15 \end{array} \right)$$



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Contradiction: $0x + 0y = 15 \implies 0 = 15$. The system is **inconsistent** (No Solution).



Summary of 2×2 Gaussian Elimination Outcomes

Final Reduced Form	Algebraic Result	Geometric Meaning	Solution Set
$\left(\begin{array}{cc c} 1 & 0 & v_1 \\ 0 & 1 & v_2 \end{array} \right)$	$x = v_1, y = v_2$	Intersecting Lines	Unique Point (v_1, v_2)
$\left(\begin{array}{cc c} 1 & a & b \\ 0 & 0 & 0 \end{array} \right)$	$0 = 0$ (Identity)	Coincident Lines	Infinitely Many
$\left(\begin{array}{cc c} 1 & a & b \\ 0 & 0 & k \end{array} \right) (k \neq 0)$	$0 = k$ (Contradiction)	Parallel Lines	No Solution $(\emptyset)$



n-Variable Systems & 3×3 Gaussian Elimination

Geometric Interpretation of 3×3 Systems

Intersections of 3 Planes in 3-Space (Section 3.3)

A 3×3 linear system represents three flat planes in 3D coordinate space (x_1, x_2, x_3) .

- **Unique Solution:** The 3 planes intersect at a single common corner point (x_1^*, x_2^*, x_3^*) .



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- **Unique Solution:** The 3 planes intersect at a single common corner point (x_1^*, x_2^*, x_3^*) .
- **Infinitely Many Solutions:** The 3 planes intersect along a continuous line or share an identical plane.
- **No Solution:** At least two planes are parallel, or three planes form a triangular prism with no single point shared by all three.



Full 3×3 Diagonalization (I_3 Target)

Target Transformation for 3×3 Systems:

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array} \right) \xrightarrow{\text{Gaussian Elimination}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & v_1 \\ 0 & 1 & 0 & v_2 \\ 0 & 0 & 1 & v_3 \end{array} \right)$$



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Column-by-Column Reduction Order:

- ❶ **Column 1:** Pivot leading 1 at (1,1), then clear zeros at (2,1) and (3,1).



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- ➌ **Column 3:** Pivot leading 1 at (3,3), then clear zeros at (1,3) and (2,3).



Example 4: 3×3 Unique Solution System

Problem Statement: Solve the system using Gaussian Elimination:

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ 2x_1 - x_2 + 3x_3 = 4 \\ 4x_1 + 5x_2 - 10x_3 = 13 \end{cases} \implies \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 3 & 4 \\ 4 & 5 & -10 & 13 \end{array} \right)$$



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Clear Column 1 below Row 1 ($-2R_1 + R_2 \rightarrow R_2$ and $-4R_1 + R_3 \rightarrow R_3$):

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & 1 & -8 \\ 0 & 1 & -14 & -11 \end{array} \right)$$



Example 4: Continued

Current Matrix:

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Interchange $R_2 \leftrightarrow R_3$ to get an easy leading 1 at $(2,2)$:

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Clear Column 2 ($-1R_2 + R_1 \rightarrow R_1$ and $3R_2 + R_3 \rightarrow R_3$):

$$\left(\begin{array}{ccc|c} \mathbf{1} & \mathbf{0} & \mathbf{15} & \mathbf{17} \\ 0 & 1 & -14 & -11 \\ \mathbf{0} & \mathbf{0} & \mathbf{-41} & \mathbf{-41} \end{array} \right)$$



Example 4: Final Pivoting & Solution

Current Matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 15 & 17 \\ 0 & 1 & -14 & -11 \\ 0 & 0 & -41 & -41 \end{array} \right)$$



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Scale Row 3 ($R_3 \div (-41) \rightarrow R_3$):

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Incomplete Diagonalization & Free Variables

Suppose Row reduction of a 3×3 system produces:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0.8 & 11 \\ 0 & 1 & 0.2 & 9 \\ 0 & 0 & 0 & 0 \end{array} \right)$$



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$$\left(\begin{array}{ccc|c} 1 & 0 & 0.8 & 11 \\ 0 & 1 & 0.2 & 9 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

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- Column 3 cannot be pivoted to a leading 1 $\implies x_3$ is a **free / arbitrary variable**.



Incomplete Diagonalization & Free Variables

Suppose Row reduction of a 3×3 system produces:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0.8 & 11 \\ 0 & 1 & 0.2 & 9 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

- Row 3 becomes $0 = 0$ (Dependent equations).
- Column 3 cannot be pivoted to a leading 1 $\implies \mathbf{x_3}$ is a **free / arbitrary variable**.
- Express basic variables in terms of x_3 :

$$x_1 + 0.8x_3 = 11 \implies \mathbf{x_1 = 11 - 0.8x_3}$$

$$x_2 + 0.2x_3 = 9 \implies \mathbf{x_2 = 9 - 0.2x_3}$$



Underdetermined Systems ($m < n$)

Underdetermined Linear Systems (Section 3.3)

A system with fewer equations than variables ($m < n$, e.g., 2 equations in 3 variables) is underdetermined.

- **Core Rule:** An underdetermined system **can never have a unique solution**.



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A system with fewer equations than variables ($m < n$, e.g., 2 equations in 3 variables) is underdetermined.

- **Core Rule:** An underdetermined system **can never have a unique solution**.
- It must have either **infinitely many solutions** or **no solution**.
- **Commercial Context:** Occurs when a firm has more decision choices than restrictive capacity guidelines, leaving operational flexibility.



Example 5: Underdetermined 2×3 System

Problem Statement: Solve the 2×3 system:

$$\begin{cases} 2x_1 - 4x_2 - 2x_3 = 6 \\ -x_1 + 2x_2 + 3x_3 = 9 \end{cases} \implies \left(\begin{array}{ccc|c} 2 & -4 & -2 & 6 \\ -1 & 2 & 3 & 9 \end{array} \right)$$



Example 5: Underdetermined 2×3 System

Problem Statement: Solve the 2×3 system:

$$\begin{cases} 2x_1 - 4x_2 - 2x_3 = 6 \\ -x_1 + 2x_2 + 3x_3 = 9 \end{cases} \implies \left(\begin{array}{ccc|c} 2 & -4 & -2 & 6 \\ -1 & 2 & 3 & 9 \end{array} \right)$$

Step 1: Scale Row 1 ($R_1 \div 2 \rightarrow R_1$):

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ -1 & 2 & 3 & 9 \end{array} \right)$$



Example 5: Underdetermined 2×3 System

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Step 1: Scale Row 1 ($R_1 \div 2 \rightarrow R_1$):

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ -1 & 2 & 3 & 9 \end{array} \right)$$

Step 2: Add Row 1 to Row 2 ($1R_1 + R_2 \rightarrow R_2$):

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & 0 & 2 & 12 \end{array} \right)$$



Example 5: Continued

Current Matrix:

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & 0 & 2 & 12 \end{array} \right)$$



Example 5: Continued

Current Matrix:

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & 0 & 2 & 12 \end{array} \right)$$

Step 3: Scale Row 2 ($R_2 \div 2 \rightarrow R_2$):

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{6} \end{array} \right)$$



Example 5: Continued

Current Matrix:

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & 0 & 2 & 12 \end{array} \right)$$

Step 3: Scale Row 2 ($R_2 \div 2 \rightarrow R_2$):

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{6} \end{array} \right)$$

Step 4: Add Row 2 to Row 1 ($1R_2 + R_1 \rightarrow R_1$):

$$\left(\begin{array}{ccc|c} \mathbf{1} & \mathbf{-2} & \mathbf{0} & \mathbf{9} \\ 0 & 0 & 1 & 6 \end{array} \right)$$



Example 5: Continued

Current Matrix:

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & 0 & 2 & 12 \end{array} \right)$$

Step 3: Scale Row 2 ($R_2 \div 2 \rightarrow R_2$):

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$$\left(\begin{array}{ccc|c} \mathbf{1} & \mathbf{-2} & \mathbf{0} & \mathbf{9} \\ 0 & 0 & 1 & 6 \end{array} \right)$$

Parametric Solution Set: From Row 2, $\mathbf{x_3 = 6}$. From Row 1, $x_1 - 2x_2 = 9 \implies \mathbf{x_1 = 2x_2 + 9}$. Variable $\mathbf{x_2}$ is arbitrary!

Selected Real-World Business Applications

Application 1: Departmental Production Capacity Allocation

Manufacturing Problem Setup

A firm manufactures 3 products (x_1, x_2, x_3). Each product requires processing across 3 departments with fixed available labor hours per week:

Department	Prod 1 (x_1)	Prod 2 (x_2)	Prod 3 (x_3)	Capacity (Hours)
Dept A	2 hrs	3.5 hrs	3 hrs	1,200 hrs
Dept B	3 hrs	2.5 hrs	2 hrs	1,150 hrs
Dept C	4 hrs	3.0 hrs	2 hrs	1,400 hrs



Application 1: System Formulation & Solution

Simultaneous Linear Model:

$$\begin{cases} 2x_1 + 3.5x_2 + 3x_3 = 1200 \\ 3x_1 + 2.5x_2 + 2x_3 = 1150 \\ 4x_1 + 3.0x_2 + 2x_3 = 1400 \end{cases} \implies \left(\begin{array}{ccc|c} 2 & 3.5 & 3 & 1200 \\ 3 & 2.5 & 2 & 1150 \\ 4 & 3.0 & 2 & 1400 \end{array} \right)$$



Application 1: System Formulation & Solution

Simultaneous Linear Model:

$$\begin{cases} 2x_1 + 3.5x_2 + 3x_3 = 1200 \\ 3x_1 + 2.5x_2 + 2x_3 = 1150 \\ 4x_1 + 3.0x_2 + 2x_3 = 1400 \end{cases} \implies \left(\begin{array}{ccc|c} 2 & 3.5 & 3 & 1200 \\ 3 & 2.5 & 2 & 1150 \\ 4 & 3.0 & 2 & 1400 \end{array} \right)$$

Gaussian Reduction Yields Diagonal Identity Form:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \mathbf{200} \\ 0 & 1 & 0 & \mathbf{100} \\ 0 & 0 & 1 & \mathbf{150} \end{array} \right)$$



Application 1: System Formulation & Solution

Simultaneous Linear Model:

$$\begin{cases} 2x_1 + 3.5x_2 + 3x_3 = 1200 \\ 3x_1 + 2.5x_2 + 2x_3 = 1150 \\ 4x_1 + 3.0x_2 + 2x_3 = 1400 \end{cases} \implies \left(\begin{array}{ccc|c} 2 & 3.5 & 3 & 1200 \\ 3 & 2.5 & 2 & 1150 \\ 4 & 3.0 & 2 & 1400 \end{array} \right)$$

Gaussian Reduction Yields Diagonal Identity Form:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \mathbf{200} \\ 0 & 1 & 0 & \mathbf{100} \\ 0 & 0 & 1 & \mathbf{150} \end{array} \right)$$

Optimal Production Schedule: Produce **200 units** of Product 1, **100 units** of Product 2, and **150 units** of Product 3 to achieve 100% labor utilization across all 3 departments.

Application 2: Commercial Coffee Blending Model

Commercial Blending Problem Setup

A coffee distributor wants to blend 3 bean grades (x_1, x_2, x_3) costing \$1.20, \$1.60, and \$1.40 per pound to fulfill a wholesale order.

- **Total Weight Target:** Order size is exactly **40,000 lbs.**



Application 2: Commercial Coffee Blending Model

Commercial Blending Problem Setup

A coffee distributor wants to blend 3 bean grades (x_1, x_2, x_3) costing \$1.20, \$1.60, and \$1.40 per pound to fulfill a wholesale order.

- **Total Weight Target:** Order size is exactly **40,000 lbs.**
- **Total Budget Target:** Order contract value is **\$57,600.**



Application 2: Commercial Coffee Blending Model

Commercial Blending Problem Setup

A coffee distributor wants to blend 3 bean grades (x_1, x_2, x_3) costing \$1.20, \$1.60, and \$1.40 per pound to fulfill a wholesale order.

- **Total Weight Target:** Order size is exactly **40,000 lbs.**
- **Total Budget Target:** Order contract value is **\$57,600.**
- **Taste Recipe Constraint:** Grade 2 (x_2) must equal twice the amount of Grade 1 (x_1)
 $\implies x_2 = 2x_1 \implies 2x_1 - x_2 + 0x_3 = 0.$



Application 2: Matrix Reduction & Blend Solution

Augmented Matrix Setup:

$$\begin{cases} x_1 + x_2 + x_3 = 40000 \\ 1.20x_1 + 1.60x_2 + 1.40x_3 = 57600 \\ 2x_1 - x_2 = 0 \end{cases} \implies \left(\begin{array}{ccc|c} 1 & 1 & 1 & 40000 \\ 1.2 & 1.6 & 1.4 & 57600 \\ 2 & -1 & 0 & 0 \end{array} \right)$$



Application 2: Matrix Reduction & Blend Solution

Augmented Matrix Setup:

$$\begin{cases} x_1 + x_2 + x_3 = 40000 \\ 1.20x_1 + 1.60x_2 + 1.40x_3 = 57600 \\ 2x_1 - x_2 = 0 \end{cases} \implies \left(\begin{array}{ccc|c} 1 & 1 & 1 & 40000 \\ 1.2 & 1.6 & 1.4 & 57600 \\ 2 & -1 & 0 & 0 \end{array} \right)$$

Applying Gaussian Elimination Step-by-Step:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 8000 \\ 0 & 1 & 0 & 16000 \\ 0 & 0 & 1 & 16000 \end{array} \right)$$



Application 2: Matrix Reduction & Blend Solution

Augmented Matrix Setup:

$$\begin{cases} x_1 + x_2 + x_3 = 40000 \\ 1.20x_1 + 1.60x_2 + 1.40x_3 = 57600 \\ 2x_1 - x_2 = 0 \end{cases} \implies \left(\begin{array}{ccc|c} 1 & 1 & 1 & 40000 \\ 1.2 & 1.6 & 1.4 & 57600 \\ 2 & -1 & 0 & 0 \end{array} \right)$$

Applying Gaussian Elimination Step-by-Step:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 8000 \\ 0 & 1 & 0 & 16000 \\ 0 & 0 & 1 & 16000 \end{array} \right)$$

Optimal Blend Mixture: Combine **8,000 lbs** of Grade 1, **16,000 lbs** of Grade 2, and **16,000 lbs** of Grade 3 beans.

Application 3: Portfolio Investment Asset Allocation

Financial Investment Parameters:

An investment manager has **\$50,000** to allocate across 3 assets: Growth Stock (x_1 , 15% return, risk 12), High-Yield Bond (x_2 , 10% return, risk 8), and Treasury Bills (x_3 , 6% return, risk 2).

- **Capital Limit:** $x_1 + x_2 + x_3 = 50,000$.



Application 3: Portfolio Investment Asset Allocation

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An investment manager has **\$50,000** to allocate across 3 assets: Growth Stock (x_1 , 15% return, risk 12), High-Yield Bond (x_2 , 10% return, risk 8), and Treasury Bills (x_3 , 6% return, risk 2).

- **Capital Limit:** $x_1 + x_2 + x_3 = 50,000$.
- **Target Annual Return (10.8% Overall = \$5,400):** $0.15x_1 + 0.10x_2 + 0.06x_3 = 5,400$.



Application 3: Portfolio Investment Asset Allocation

Financial Investment Parameters:

An investment manager has **\$50,000** to allocate across 3 assets: Growth Stock (x_1 , 15% return, risk 12), High-Yield Bond (x_2 , 10% return, risk 8), and Treasury Bills (x_3 , 6% return, risk 2).

- **Capital Limit:** $x_1 + x_2 + x_3 = 50,000$.
- **Target Annual Return (10.8% Overall = \$5,400):** $0.15x_1 + 0.10x_2 + 0.06x_3 = 5,400$.
- **Target Average Risk Score (7.6 Overall = 380,000 total score):**
 $12x_1 + 8x_2 + 2x_3 = 380,000$.



Application 3: Portfolio System Solution

Matrix System:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 50000 \\ 0.15 & 0.10 & 0.06 & 5400 \\ 12 & 8 & 2 & 380000 \end{array} \right)$$



Application 3: Portfolio System Solution

Matrix System:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 50000 \\ 0.15 & 0.10 & 0.06 & 5400 \\ 12 & 8 & 2 & 380000 \end{array} \right)$$

Gaussian Reduction Output:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \mathbf{20000} \\ 0 & 1 & 0 & \mathbf{15000} \\ 0 & 0 & 1 & \mathbf{15000} \end{array} \right)$$



Application 3: Portfolio System Solution

Matrix System:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 50000 \\ 0.15 & 0.10 & 0.06 & 5400 \\ 12 & 8 & 2 & 380000 \end{array} \right)$$

Gaussian Reduction Output:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \mathbf{20000} \\ 0 & 1 & 0 & \mathbf{15000} \\ 0 & 0 & 1 & \mathbf{15000} \end{array} \right)$$

Optimal Portfolio Allocation:

Invest **\$20,000** in Growth Stocks, **\$15,000** in Bonds, and **\$15,000** in Treasury Bills to achieve the exact target return and risk profile.



Discussion: Sensitivity Analysis

Discussion

Managerial Sensitivity Questions

- 1 In the coffee blending model, if the supplier increases the price of Grade 2 beans from \$1.60 to \$1.75 per pound, can the distributor keep the exact same blend ratio without increasing the \$57,600 total budget?
- 2 In the departmental production model, what happens if Department B experiences a machine breakdown that reduces capacity from 1,150 hours to 900 hours? Will the system remain consistent?



Summary & Homework Assignment

Summary & Practice Homework Assignment

Lecture 04 Summary

- Gaussian Elimination provides an efficient, row-operation algorithm for solving simultaneous systems.
- Full diagonalization (I_n) yields unique solutions; zero rows indicate dependent or inconsistent conditions.
- Real-world problems (blending, portfolio, manufacturing) are solved by mapping requirements to matrix arrays.

Homework Problems (Budnick, 4th Ed)

- **Section 3.2:** Exercises 1, 5, 9, 15, 22
- **Section 3.3:** Exercises 3, 7, 11, 15, 19
- **Section 3.4:** Exercises 2, 7, 8

Practice these textbook problems to master matrix pivoting ahead of **Quiz 1!**

