



Lecture 09

Fundamentals of Business Mathematics

MATH-114 • BBA 2k26

Dr. Rizwan Jahangir

NUST Business School

National University of Sciences & Technology

Lecture 09 Outline: Break-Even Models & Quadratic Functions

Part 1: Break-Even Models

- Break-even point
- Assumptions of the model
- Break-even analysis
- Example 12: Smoke detectors
- Practice exercise

Part 2: Quadratic Functions

- Mathematical form
- Graphical representation
- Concavity, intercepts, and vertex
- Roots by factoring and quadratic formula
- Sketching quadratic functions
- Quadratic estimating function: NBA salaries



Break-Even Models

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- Output below the break-even level results in a loss, while output above it results in a profit, under the model assumptions.
- Break-even analysis is useful as a planning tool when considering a new product, service expansion, or business venture.



Assumptions of the Break-Even Model

The model considered in this section assumes that both total cost and total revenue are linear.

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- ④ **Selling price per unit is constant.**
- ⑤ The **price per unit exceeds the variable cost per unit.**

If price per unit were less than variable cost per unit, the firm would lose money on every unit produced and sold, so a break-even condition could not exist.



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Primary objective: determine the break-even point.

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- ④ Solve for x . The resulting value is the break-even level of output, denoted by x_{BE} .



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- 3 set $P(x) = 0$;
- 4 solve for x_{BE} .

Both approaches give the same break-even output because $P(x) = 0$ is equivalent to $R(x) = C(x)$.

Example 12: Smoke Detectors

- ~ **Problem.** A group of engineers plans to produce smoke detectors.
- Variable cost per detector: \$22.50.
 - Fixed costs: \$250,000.
 - Selling price: \$30 per detector.
- (a) Determine the number of detectors that must be sold to break even.
(b) If 30,000 detectors are sold, determine the expected profit.

Let x be the number of smoke detectors produced and sold.



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The firm must sell approximately **33,333.33 smoke detectors** to break even under the assumed price and cost parameters.

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A negative profit means a loss. If the price, cost, and demand estimates hold, the firm expects a **\$25,000 loss** at an output of 30,000 detectors.



Practice Exercise: Check the Break-Even Point

Discussion

At the break-even point from Example 12,

$$x_{BE} = 33,333.33 \text{ units.}$$

Verify that total revenue and total cost are both approximately \$1,000,000, taking rounding into account.



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$$C(x_{BE}) = 22.50(33,333.33) + 250,000 \approx 750,000 + 250,000 = 1,000,000.$$



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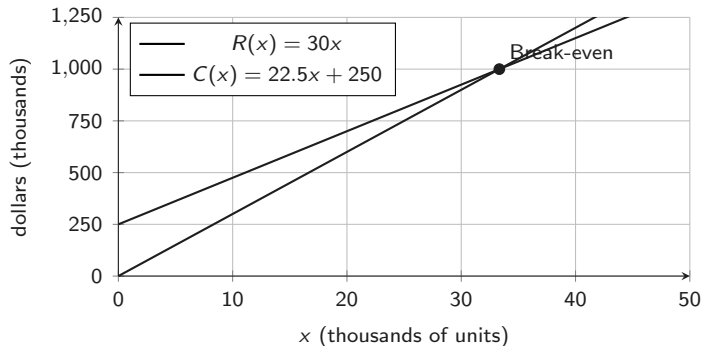
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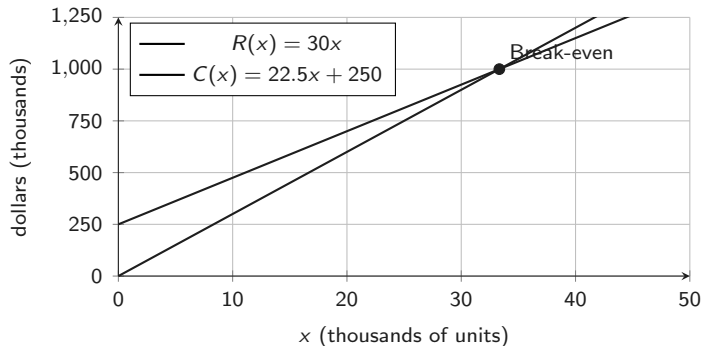
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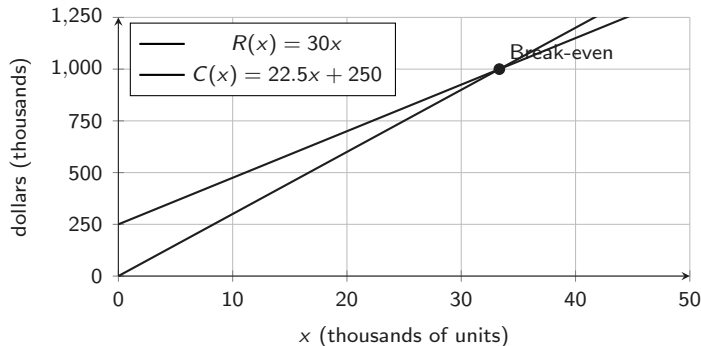
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- To the **right**:

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so there is a profit.



Quadratic Functions and Their Characteristics

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- Many real-world phenomena do not behave linearly.
- Quadratic functions provide a simple nonlinear model whose graph has a characteristic parabolic shape.
- We will focus on the algebraic and graphical characteristics needed to recognize and sketch quadratic functions.



Mathematical Form of a Quadratic Function

A quadratic function has the form

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- b and c may be any real numbers.
- The coefficient a determines the orientation of the parabola.



Example 1: Recognizing Quadratic Functions

Determine which functions are quadratic.

① $y = 5x^2$

② $u = -10v^2 - 6$

③ $y = 7x - 2$

④ $g = -6$

⑤ $y = x^2 + 10x$

⑥ $y = 6x^2 - 40x + 15$

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Quadratic: (1), (2), (5), and (6).

Not quadratic: (3), (4), and (7).

The reason is structural: a quadratic must have a nonzero coefficient on the second-degree term and cannot contain a third-degree term.



Practice Exercise: Recognize Quadratic Functions

Which of the following are quadratic functions?

① $y = (50 - 0.001x)x$

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Answer: (1) and (2).

Tip: Expand the expression when necessary and check the highest power of the variable.



Graphical Representation: Parabolas

All quadratic functions of the form

$$y = ax^2 + bx + c, \quad a \neq 0,$$

graph as **parabolas**.

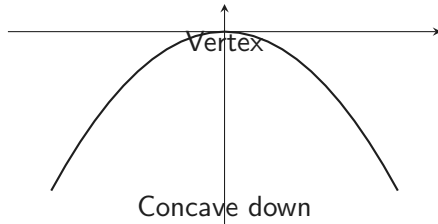
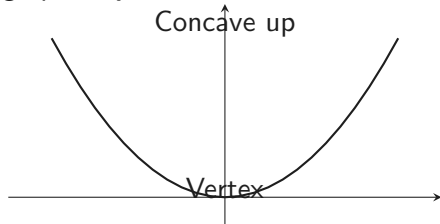


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For $y = ax^2 + bx + c$, the vertex is

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a} \right) \right).$$



Example 2: Determining Concavity

For each quadratic function, determine whether its graph is concave up or concave down.

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- $a = 5 > 0 \Rightarrow$ concave up.
- $a = -10 < 0 \Rightarrow$ concave down.
- $a = 1 > 0 \Rightarrow$ concave up.
- $a = 6 > 0 \Rightarrow$ concave up.



The y -Intercept of a Quadratic Function

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Example 2 functions:

- $y = 5x^2$: $(0, 0)$
- $u = -10v^2 - 6$: $(0, -6)$
- $y = x^2 + 10x$: $(0, 0)$
- $y = 6x^2 - 40x + 15$: $(0, 15)$



The x -Intercepts and Roots

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The number of real roots determines how many times the parabola meets the x -axis.



Example 4: Determining Roots by Factoring

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Set each factor equal to zero:

$$2x = 0 \Rightarrow x = 0, \quad 3x - 1 = 0 \Rightarrow x = \frac{1}{3}.$$



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The parabola has x -intercepts at

$$(0, 0) \quad \text{and} \quad \left(\frac{1}{3}, 0\right).$$



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Thus,

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The quadratic function

$$y = x^2 + 6x + 9$$

has one x -intercept,

$$(-3, 0),$$

because the root is repeated.



Example 5: Determining Roots Using the Quadratic Formula

Given

$$f(x) = x^2 - x - 3.75,$$

the x -intercepts occur when

$$x^2 - x - 3.75 = 0.$$

Thus $a = 1$, $b = -1$, and $c = -3.75$.



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Substitute:

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-3.75)}}{2}.$$



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$$x = \frac{1 \pm \sqrt{16}}{2} = \frac{1 \pm 4}{2}.$$



Example 5: Intercepts from the Roots

From

$$x = \frac{1 \pm 4}{2},$$

we obtain

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When factoring is not convenient, the quadratic formula provides a systematic way to find the real roots, when they exist.



The Vertex of a Quadratic Function

For

$$f(x) = ax^2 + bx + c,$$

the x -coordinate of the vertex is

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If the parabola has two x -intercepts, the x -coordinate of the vertex lies midway between them.



Example 6: Locating the Vertex

Find the vertex of

$$f(x) = 2x^2 + 3x - 20.$$

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This agrees with the formula $x_V = -b/(2a)$.



Example 7: Sketching a Quadratic Function

Consider

$$f(x) = 3x^2 + 6x - 45.$$

To sketch the parabola, determine four key attributes:

- 1 concavity,
- 2 y -intercept,
- 3 x -intercepts,
- 4 vertex.



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y -intercept:

$$f(0) = -45 \Rightarrow (0, -45).$$



Example 7: Finding the x -Intercepts

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Hence,

$$x = 3 \quad \text{or} \quad x = -5.$$

Therefore the x -intercepts are

$$(3, 0) \quad \text{and} \quad (-5, 0).$$



Example 7: Finding the Vertex and Sketch

Because there are two x -intercepts, the x -coordinate of the vertex is their midpoint:

$$x_V = \frac{-5 + 3}{2} = -1.$$



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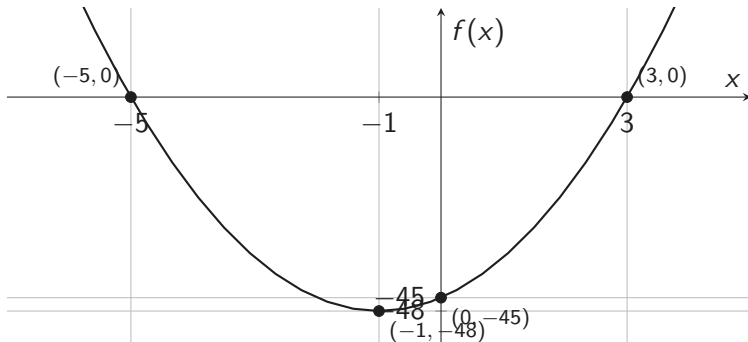
Thus

$$V = (-1, -48).$$

A reasonable sketch is obtained from the concavity, y -intercept, two x -intercepts, and vertex.



Example 7: Graph of $f(x) = 3x^2 + 6x - 45$



Practice Exercise: Complete the Quadratic Characteristics

Given

$$f(x) = 2x^2 - x - 15,$$

determine:

- 1 the concavity;
- 2 the y -intercept;
- 3 the x -intercept(s);
- 4 the location of the vertex.



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determine:

- ① the concavity;
- ② the y -intercept;
- ③ the x -intercept(s);
- ④ the location of the vertex.

Answer:

- Concavity: up.
- y -intercept: $(0, -15)$.
- x -intercepts: $(-2.5, 0)$ and $(3, 0)$.
- Vertex: $(0.25, -15.125)$.



Example 8: Quadratic Estimating Function for NBA Salaries

^Average NBA player salaries were approximately:

- 1981: \$190,000
- 1985: \$310,000
- 1988: \$600,000

We seek a quadratic estimating function

$$s = f(t) = at^2 + bt + c,$$

where s is average salary in thousands of dollars and t is the number of years since 1981.



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Three data points are needed because there are three parameters a , b , and c .



Example 8: Determining the Parameters

Substitute the three data points into

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Therefore,

$$c = 190,$$

and

$$16a + 4b = 120, \quad 49a + 7b = 410.$$



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Therefore,

$$c = 190,$$

and

$$16a + 4b = 120, \quad 49a + 7b = 410.$$

Solving gives

$$a = 9.524, \quad b = -8.096.$$



Example 8: The Estimating Function

The quadratic estimating function is

$$s = f(t) = 9.524t^2 - 8.096t + 190.$$



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This function is a parabola passing through the three salary data points.

The time variable was redefined as years since 1981:

$$1981 \Rightarrow t = 0, \quad 1985 \Rightarrow t = 4, \quad 1988 \Rightarrow t = 7.$$

This makes the data easier to use in the quadratic model.



Practice Exercise: Estimate the 1995 NBA Salary

Using

$$s = f(t) = 9.524t^2 - 8.096t + 190,$$

estimate the average salary in 1995.



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Since s is measured in thousands of dollars,

$$s(14) \approx \$1,943,360.$$



Lecture 09: Summary

Break-Even Models

- Break-even means zero profit.
- At break-even, $R(x) = C(x)$.
- Equivalently, $P(x) = 0$.
- The model assumes linear cost and revenue relationships and constant unit price/variable cost.

Quadratic Functions

- $f(x) = ax^2 + bx + c$, $a \neq 0$.
- $a > 0$: concave up; $a < 0$: concave down.
- y-intercept: $(0, c)$.
- x-intercepts come from $ax^2 + bx + c = 0$.
- Vertex: $(-\frac{b}{2a}, f(-\frac{b}{2a}))$.
- A quadratic graph is a parabola.

